

## Natural Variations in the Analytical Form of the Raindrop Size Distribution

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(Manuscript received 22 February 1983, in final form 6 June 1983)

### ABSTRACT

Empirical analyses are shown to imply variation in the shape or analytical form of the raindrop size distribution consistent with that observed experimentally and predicted theoretically. These natural variations in distribution shape are demonstrated by deriving relationships between pairs of integral rainfall parameters using a three parameter gamma drop size distribution and comparing the expressions with empirical results. These comparisons produce values for the size distribution parameters which display a systematic dependence of one of the parameters on another between different rainfall types, as well as from moment to moment within a given rainfall type. The implications of this finding are explored in terms of the use of a three-parameter gamma distribution in dual-measurement techniques to determine rainfall rate.

### 1. Introduction

The analysis of rainfall data has assumed traditionally that the raindrop size distribution (DSD) has exponential form; that is, with  $N(D)$  ( $\text{m}^{-3} \text{cm}^{-1}$ ) equal to the number of raindrops per unit volume per unit size interval having equivolume spherical diameter  $D$  (cm), then

$$N(D) = N_0 \exp(-\Lambda D) \quad (0 \leq D \leq D_{\max}), \quad (1)$$

where  $N_0$  ( $\text{m}^{-3} \text{cm}^{-1}$ ) and  $\Lambda$  ( $\text{cm}^{-1}$ ) are parameters of the distribution and  $D_{\max}$  is the maximum drop diameter. This form was originally found by Marshall and Palmer (1948) who also suggested that  $\Lambda$  varied with rainfall rate  $R$  ( $\text{mm h}^{-1}$ ) as  $\Lambda = 41R^{-0.21}$  and that  $N_0$  had the constant value  $N_0 = 8 \times 10^4 \text{m}^{-3} \text{cm}^{-1}$ . A similar analysis of the drop size spectra of Laws and Parsons (1943) reveals that their data can also be represented closely by exponential form but with  $\Lambda = 38R^{-0.20}$  and  $N_0$  weakly dependent on rainfall rate through  $N_0 = 5.1 \times 10^4 R^{-0.03}$ .

There exist many other experimental results which suggest that Eq. (1) is a very good approximation to the raindrop size distribution in conditions similar to those which apply to the Marshall and Palmer and Laws and Parsons spectra, viz., where sufficient averaging in space and/or time is performed. However, Waldvogel (1974) and Donnadieu (1982) have shown that large and sudden changes in  $N_0$  can occur from moment to moment within a given rainfall type. In these situations it is therefore necessary to account for deviations of  $N_0$  from the constant value found by Marshall and Palmer, or the weak dependence of  $N_0$  on  $R$  implied by the Laws and Parsons spectra. Furthermore, these variations in  $N_0$  are independent of those that occur in  $\Lambda$  from moment to moment so

that to describe rainfall parameters accurately it is necessary to specify both of these two independent DSD variables. The consequences of this conclusion have been explored in depth in many previous investigations (e.g., Atlas and Ulbrich, 1974; Goldhirsh and Katz, 1974; Hauser and Amayenc, 1983; Seliga and Bringi, 1976; Seliga *et al.*, 1981; Ulbrich and Atlas, 1975, 1977, 1978, 1984; Ulbrich, 1981) particularly with regard to its implications in remote measurement techniques for determining rainfall rate. These investigations have shown that a significant improvement in measurement accuracy is obtained when pairs of remote measurables are used to determine the two DSD parameters and thus all precipitation parameters defined in terms of them. This improvement in accuracy is a consequence of the fact that such dual-measurement techniques allow for natural, independent variations in both of the DSD parameters and alleviate the need for the introduction of empirical relations.

In some cases, variations of the DSD from exponentiality [i.e., from the form given by Eq. (1)] can have important effects on the values of rainfall parameters deduced from dual-measurement techniques (Ulbrich and Atlas, 1977; Ulbrich, 1981). These effects are particularly pronounced in the case of rainfall rates deduced from differential reflectivity measurements. For this technique Ulbrich and Atlas (1984) have shown that further improvement in measurement accuracy can be achieved if the DSD is assumed to be a gamma distribution, i.e., of the form

$$N(D) = N_0 D^\mu \exp(-\Lambda D), \quad 0 \leq D \leq D_{\max}, \quad (2)$$

where the exponent  $\mu$  can have any positive or negative value and the coefficient  $N_0$  now has the units  $\text{m}^{-3} \text{cm}^{-1-\mu}$ . Additional support for the DSD having this

form in differential reflectivity measurements is found in the experimental results of Goddard and Cherry (1984) and Bringi *et al.* (1984).

Since Eq. (2) involves three DSD parameters, then this form of the distribution can be used in dual-measurement techniques only if *a priori* knowledge exists concerning the behavior of one of the parameters or of a relationship between two of them. Alternatively, a third remote measurable could be introduced thereby enabling simultaneous determination of all three DSD parameters. Atlas *et al.* (1984) describe a simulation of such a three-measurable technique using experimental raindrop size spectra for which they obtain virtually perfect agreement between calculated and observed rainfall rates.

In this work further justification is presented for employing a DSD having the form given by Eq. (2) by demonstrating that previous empirical analyses of other investigators imply that variations in the form or shape of the DSD occur commonly in nature. It is further shown that the DSD variations found between different rainfall types are similar to those found from moment to moment within a given rainfall type. Finally, methods are outlined which would permit these variations to be incorporated into dual-measurement methods thereby alleviating the necessity of introducing a third remote measurable.

## 2. Theoretical relations: Gamma distribution

The gamma distribution has been suggested as an appropriate form for the distribution of droplets in clouds by Khragian *et al.* (1952), of aerosols by Levin (1961), and of precipitation particles by Sulakvelidze (1969) and Sulakvelidze and Dadali (1971). Some of the relations shown in this work are similar to those presented in the latter reference.

### a. Gamma drop size distribution parameters

When employing a form like Eq. (2) for the DSD it is usually assumed that the product  $\Lambda D_{\max}$  is large. In such a case the median volume diameter  $D_0$  (cm) can be expressed uniquely in terms of  $\Lambda$ . For an exponential distribution ( $\mu = 0$ ) Sekhon and Srivastava (1970) have shown that  $\Lambda D_0 = 3.672$  when  $\Lambda D_{\max} \rightarrow \infty$  and is within 2% of this limiting value for all  $\Lambda D_{\max} \geq 10$  (i.e.,  $D_{\max}/D_0 \geq 2.5$ ). For values of  $\mu$  different from zero the relationship between  $\Lambda$ ,  $D_0$  and  $D_{\max}$  is determined from the definition of  $D_0$ , i.e.,

$$2 \int_0^{D_0} D^3 N(D) dD = \int_0^{D_{\max}} D^3 N(D) dD, \quad (3)$$

which can also be written

$$2\gamma(4 + \mu, \Lambda D_0) = \gamma(4 + \mu, \Lambda D_{\max}), \quad (4)$$

where  $\gamma(a, x)$  is the incomplete gamma function. A plot of the relationship between  $\Lambda D_0$  and  $\Lambda D_{\max}$  (or

$D_{\max}/D_0$ ) as defined by Eq. (4) is shown in Fig. 1 for values of  $-2 \leq \mu \leq 3$ . It is seen that the value of  $\Lambda D_{\max}$  at which  $\Lambda D_0$  is close to its limiting value depends on  $\mu$ , but for  $\mu \geq -2$  it may be concluded that  $\Lambda D_0$  is very close to its limiting value at all  $\mu$  provided  $D_{\max}/D_0 \geq 2.5$ . It can be further shown that these limiting values are given by the approximate expression

$$\Lambda D_0 = 3.67 + \mu \quad (5)$$

which is accurate to within 0.5% for all  $\mu > -3$ . An even better approximation is  $\Lambda D_0 = 3.67 + \mu + 10^{-0.3(\mu+9)}$  which gives limiting values of  $\Lambda D_0$  which are within 0.02% of their exact values at all  $\mu > -3$ . In the remainder of this work it is assumed that  $\Lambda D_0$  is given by Eq. (5) and that the upper limits on all integral parameters can be allowed to tend to infinitely large values.

The shape of the DSD as a function of  $D$  is determined solely by the exponent  $\mu$ . That is, for positive values of  $\mu$  the DSD is concave down on a plot of  $\ln[N(D)]$  versus  $D$ , has narrow breadth and falls rapidly to zero as  $D \rightarrow 0$ . For negative values of  $\mu$  the DSD is concave upward on a semilogarithmic plot and has large breadth with increased numbers of drops at both small and large diameters. These DSD shapes are illustrated in Fig. 2 for three distributions all having the same liquid water content and median volume diameter but with differing values of  $\mu$ . It is apparent that the form given by Eq. (2) allows for a wide variety of DSD shapes and breadths. That such a variety of forms occurs in nature will be demonstrated in Section 3.

One of the advantages of assuming an exponential form for the DSD is that both of the DSD parameters can be determined graphically on a plot of  $\ln[N(D)]$  versus  $D$ . Although not all three of the parameters in Eq. (2) can be determined on a single plot, two of them can be interpreted graphically in a manner similar to that for the exponential distribution. On a plot of  $\ln[N(D)]$  versus  $D$  the portion of the gamma distri-

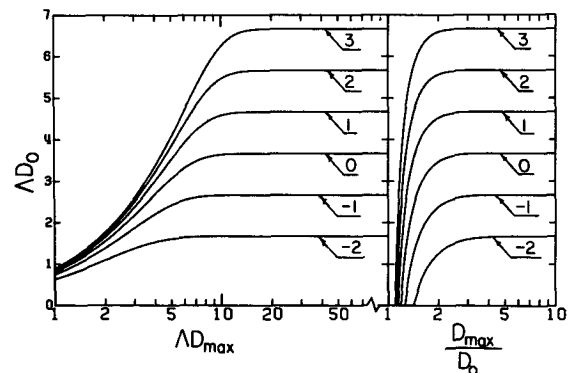


FIG. 1. The product  $\Lambda D_0$  as a function of  $\Lambda D_{\max}$  and  $D_{\max}/D_0$  for a gamma raindrop size distribution having the form  $N(D) = N_0 D^\mu \times \exp(-\Lambda D)$  ( $0 \leq D \leq D_{\max}$ ). Each of the curves is labeled with the value of the exponent  $\mu$  to which it corresponds.

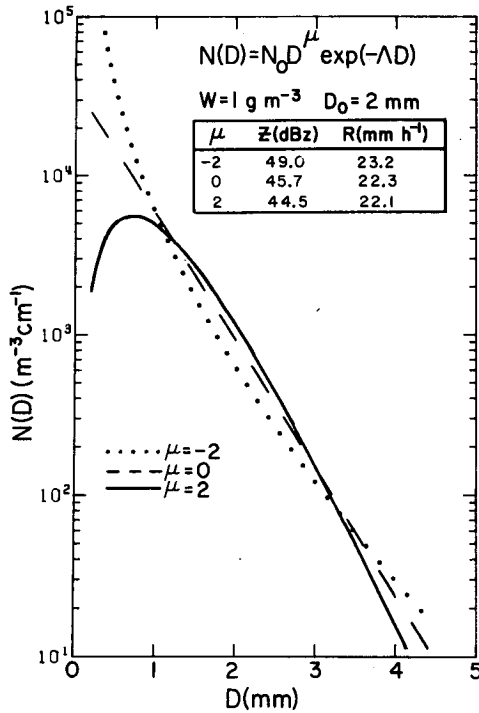


FIG. 2. Examples of the gamma raindrop size distribution for  $\mu = -2, 0$  and  $2$  and with liquid water content  $W = 1 \text{ g m}^{-3}$  and median volume diameter  $D_0 = 2 \text{ mm}$ . The inset table shows the corresponding values of radar reflectivity factor (Rayleigh approximation) and rainfall rate.

bution at diameters large compared to  $D_0$  is linear and has slope equal in magnitude to  $\Lambda$ . A second plot of  $\ln[N(D)]$  versus  $\ln(D)$  approaches linearity at diameters small compared to  $D_0$  with slope equal to  $\mu$ . These two plots could therefore, in principle, be used to derive estimates of  $\Lambda$  and  $\mu$  which together define the representative diameter  $D_0$  and the shape of the DSD. However, in practice such a procedure would of course not be used. A simpler procedure will be described later which uses the experimental size spectrum data and produces fits of the gamma DSD which are better than those obtained by graphical means or by least squares.

*b. Integral rainfall parameters*

For the purposes of the present work it will be assumed that all of the integral rainfall parameters of interest can be represented by the form

$$P = a_p \int_0^\infty D^p N(D) dD \quad (6)$$

which, when Eqs. (2) and (5) are used, takes the form

$$P = a_p \frac{\Gamma(p + \mu + 1)}{(3.67 + \mu)^{p+\mu+1}} N_0 D_0^{p+\mu+1}, \quad (7)$$

where  $\Gamma(x) = \gamma(x, \infty)$  is the complete gamma function.

Table 1 lists the values of the coefficient  $a_p$  and exponent  $p$  corresponding to the integral parameters radar reflectivity factor (Rayleigh approximation)  $Z$  ( $\text{mm}^6 \text{ m}^{-3}$ ), rainfall rate  $R$  ( $\text{mm h}^{-1}$ ), liquid water content  $W$  ( $\text{g m}^{-3}$ ), microwave attenuation  $A$  ( $\text{dB km}^{-1}$ ), and optical extinction  $\Sigma$  ( $\text{km}^{-1}$ ). In construction of this table it has been assumed that power law approximations to the particle fallspeed  $v(D)$  and microwave total attenuation cross section  $Q_t(D)$  are valid. Therefore, the former has the form  $v(D) = 17.67D^{0.67}[v(D) \text{ (m s}^{-1})]$ ,  $D$  (cm)] as suggested by Atlas and Ulbrich (1977) and the latter is of the form  $Q_t(D) = C_A D^n [Q_t(D) \text{ (cm}^2)]$ ,  $D$  (cm)]. Of course,  $C_A$  and  $n$  are functions of wavelength and their values can be found in Atlas and Ulbrich (1974).

To illustrate the effects of variations in  $\mu$  on these integral rainfall parameters, the fractional contribution in size categories of width  $\Delta(D/D_0) = 0.5$  has been computed as a function of  $\mu$  for  $p = 2, 3, 4$  and  $6$ . The first two of these examples correspond to  $P = \Sigma$  and  $W$  (optical extinction and liquid water content), respectively. The fourth corresponds to  $P = Z$  (reflectivity factor). The third is an approximation to  $P = A$  (microwave attenuation) in the X-band. The results are shown in Fig. 3(a)–(d) for  $p = 2, 3, 4$  and  $6$ , respectively. The dashed and solid histograms correspond to  $\mu = -2$  and  $2$ , respectively. These results show that for  $p < 6$  the fractional contribution to integral parameters for  $\mu = -2$  is greater in both the small and large size categories than for  $\mu = 2$ . Clearly then, these integral parameter distributions are increased in breadth at both ends of the diameter scale. However, for  $p = 6$  the increase is accomplished primarily at the large diameters.

One of the assumptions implicit in Eq. (6) is that the lower and upper limits of integration can be chosen as  $D_{\min} = 0$  and  $D_{\max} \rightarrow \infty$ , respectively. Sekhon and Srivastava (1970) have investigated the accuracy of assuming  $D_{\max} \rightarrow \infty$  for integral parameters defined in terms of an exponential distribution. They find that the error introduced by allowing  $D_{\max}$  to be finite is negligible for all  $p \leq 6$  provided  $D_{\max}/D_0 \geq 3.2$ . Inspection of Fig. 3 shows that this same conclusion

TABLE 1. Coefficients  $a_p$  and exponents  $p$  in expressions of the form

$$P = a_p \int_0^\infty D^p N(D) dD = a_p \frac{\Gamma(p + \mu + 1)}{(3.67 + \mu)^{p+\mu+1}} N_0 D_0^{p+\mu+1}$$

for integral rainfall parameters  $P$  which use the gamma raindrop size distribution for  $N(D)$ .

$P$	$p$	$a_p$
$Z$	6	$10^6 \text{ mm}^6 \text{ cm}^{-6}$
$W$	3	$0.524 \text{ g cm}^{-3}$
$\Sigma$	2	$0.157 \text{ km}^{-1} \text{ m}^3 \text{ cm}^{-2}$
$R$	3.67	$33.31 \text{ mm h}^{-1} \text{ m}^3 \text{ cm}^{-3.67}$
$A$	$n$	$0.4343 C_A \text{ dB km}^{-1} \text{ m}^3 \text{ cm}^{-n}$

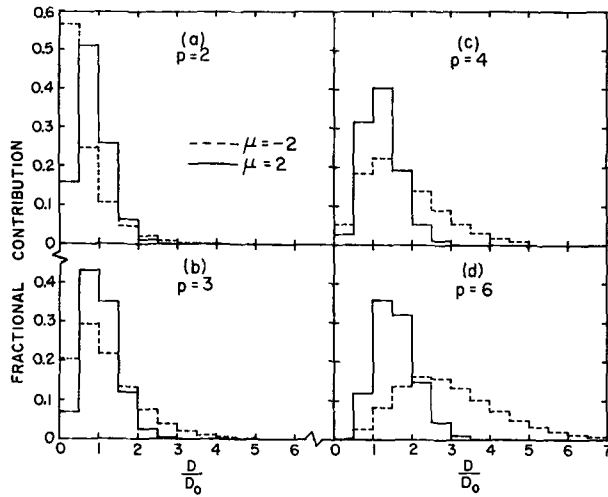


FIG. 3. Fractional contribution to integral rainfall parameters having  $p = 2, 3, 4$  and  $6$  (a)–(d), respectively in size categories of width  $\Delta(D/D_0) = 0.5$  for gamma drop size distributions with  $\mu = -2$  and  $2$ .

applies to the present work except for the case where  $\mu = -2, p = 6$ . For this case it is evident that one must require  $D_{max}/D_0 > 5$  for the error to be small. However, it will be seen presently that  $\mu < 0$  usually corresponds to the specific case of orographically induced rainfall with small  $D_0$ . Consequently, the assumption of large  $D_{max}/D_0$  will not be restrictive in most meteorological situations. Ulbrich and Atlas (1984) have shown that similar conclusions can be drawn for the differential reflectivity.

The effect of taking the lower limit of integration to be zero can also be deduced from Fig. 3. It is clear that if  $D_{min} > 0$  then the error introduced by assuming  $D_{min} = 0$  will be most pronounced at small  $p$  and  $\mu < 0$ . However, as before, this case usually corresponds to rainfall composed of large numbers of very small drops so that the assumption that  $D_{min} = 0$  should be a good approximation. In the remainder of this work it will be assumed that  $D_{min}/D_0$  is small and  $D_{max}/D_0$  is large so that all integral parameters can be represented by Eq. (7). More quantitative information concerning other assumptions for the limits of integration can be obtained from mathematical tables of the incomplete gamma function.

*c. Distribution shape parameters*

Several parameters have been introduced by previous workers which serve as measures of the breadth or shape of the DSD and which can be expressed in terms of the exponent  $\mu$ . One of them is the factor  $G$  introduced by Bartnoff and Atlas (1951) which is a dimensionless third moment of the mass spectrum, i.e.,

$$G = \frac{\int_0^\infty D^6 N(D) dD}{D_0^3 \int_0^\infty D^3 N(D) dD} = \frac{\Gamma(7 + \mu)}{(\Delta D_0)^3 \Gamma(4 + \mu)}$$

$$= \frac{(6 + \mu)(5 + \mu)(4 + \mu)}{(3.67 + \mu)^3}, \tag{8}$$

where  $\Gamma(x) = \gamma(x, \infty)$  is the complete gamma function. Another similar breadth parameter, defined in terms of the mass-weighted average diameter  $D_m$  rather than  $D_0$ , is

$$G' = \frac{\int_0^\infty D^6 N(D) dD}{D_m^3 \int_0^\infty D^3 N(D) dD} = \frac{\Gamma(7 + \mu)}{(\Delta D_m)^3 \Gamma(4 + \mu)}$$

$$= \frac{(6 + \mu)(5 + \mu)}{(4 + \mu)^2}, \tag{9}$$

where

$$D_m = \frac{\int_0^\infty D^4 N(D) dD}{\int_0^\infty D^3 N(D) dD} = \frac{4 + \mu}{\Lambda} = \left( \frac{4 + \mu}{3.67 + \mu} \right) D_0. \tag{10}$$

The advantages of using  $D_m$  rather than  $D_0$  have been described by Weber (1976). The use of  $D_m$  in analyses of precipitation data is more convenient computationally and avoids interpolation of particle size spectra to find  $D_0$ . In any event,  $D_m$  is a very good approximation to  $D_0$  for all  $\mu \geq -2$  as is evident in the plot of  $D_m/D_0$  versus  $\mu$  in Fig. 4; also shown are the dependences on  $\mu$  of  $G, G'$ , and  $\sigma_m^2/D_0^2$ , where  $\sigma_m^2$  is the variance of the mass spectrum with respect to  $D_m$  and is given by

$$\sigma_m^2 = \frac{\int_0^\infty (D - D_m)^2 D^3 N(D) dD}{\int_0^\infty D^3 N(D) dD} = \frac{4 + \mu}{\Lambda^2}$$

$$= \frac{4 + \mu}{(3.67 + \mu)^2} D_0^2. \tag{11}$$

It is clear that for  $\mu < 3, G, G'$  and  $\sigma_m^2/D_0^2$  are equally sensitive measures of DSD breadth with large values corresponding to negative  $\mu$ . For  $\mu > 3$  these parameters are less sensitive to changes in  $\mu$ ; in this range of values of  $\mu$  the DSD has such narrow breadth that further increases in  $\mu$  have a less pronounced effect on integral rainfall parameters. Nevertheless, these measures of DSD shape or breadth are more sensitive than other more commonly used parameters such as the variance of the number distribution with respect to the number-

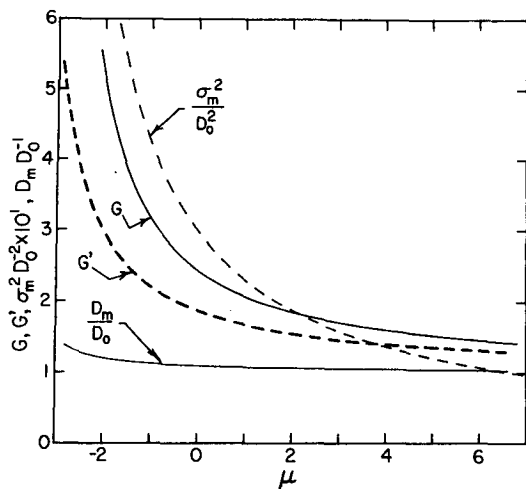


FIG. 4. Two dimensionless third moments of the mass distribution  $G$  and  $G'$ , the dimensionless variance of the mass distribution  $\sigma_m^2/D_0^2$  and the ratio of the mass-weighted average diameter to the median volume diameter  $D_m/D_0$ , as functions of the exponent  $\mu$  in the gamma raindrop size distribution. Each curve was constructed using the assumption that the product  $\Delta D_{\max} \rightarrow \infty$ .

weighted average diameter. It can be shown that this parameter is much more slowly varying with  $\mu$  than  $\sigma_m^2$  because it places greater weight on the portion of the spectrum containing small diameter drops, whereas  $\sigma_m^2$  places greater weight on the central portion of the spectrum.

Joss and Gori (1978) define DSD shape parameters in terms of integral parameters  $P$  and  $Q$  as

$$S(PQ) = \frac{\left| \frac{D(P) - D(Q)}{D(P) + D(Q)} \right|_{\text{observed}}}{\left| \frac{D(P) - D(Q)}{D(P) + D(Q)} \right|_{\text{exponential}}}, \quad (12)$$

where

$$D(P) = \frac{\int_0^{\infty} D^p N(D) dD}{\int_0^{\infty} D^{p-1} N(D) dD}. \quad (13)$$

A similar definition follows from Eq. (13) for  $D(Q)$  in terms of a corresponding exponent  $q$ . Joss and Gori calculate these shape factors for observed drop size spectra with  $P$  and  $Q$  chosen from  $Z$ ,  $W$  and  $\Sigma$  (as defined in Table 1) and an additional integral parameter designated  $R^*$  and defined by Eq. (6) with  $p = 4$ . The pairs chosen are  $(P, Q) = (Z, \Sigma)$ ,  $(Z, R^*)$ ,  $(R^*, W)$  and  $(W, \Sigma)$  corresponding to  $(p, q) = (6, 2)$ ,  $(6, 4)$ ,  $(4, 3)$  and  $(3, 2)$ , respectively. If it is assumed that the observed DSD can be represented by a gamma distribution so that Eq. (3) can be used for  $N(D)$  in Eq. (12) and (13), then these shape factors can be shown to have the form

$$S(PQ) = \frac{p + q}{p + q + 2\mu}. \quad (14)$$

It is seen that the  $S(PQ)$  are given solely in terms of the exponent  $\mu$  and that values of  $S(PQ) < 1$  correspond to concave downward distributions in agreement with the observations of Joss and Gori. Values of  $S(PQ) > 1$  correspond to concave upward distributions and such behavior is evident in a few instances in the work of Gori and Geotis (1981). It can be shown that the behavior as a function of  $\mu$  of each of the  $S(PQ)$  used by Joss and Gori (1978) is very similar to that shown in Fig. 4 for  $G$ ,  $G'$ , and  $\sigma_m^2/D_0^2$  when the latter parameters are normalized to their values for  $\mu = 0$ . In fact, there is little difference between all of the  $S(PQ)$  for  $\mu > 0$ . For  $\mu < 0$ ,  $S(R^*W)$  and  $S(ZR^*)$  are very similar to the normalized  $G'$  and  $G$ , respectively. It may be concluded then that the  $S(PQ)$  introduced by Joss and Gori and the shape parameters defined earlier by other workers and displayed in Fig. 4 serve equally well in characterizing the shape of the DSD when it is assumed that it can be represented by a gamma distribution.

### 3. Evidence for DSD shape effects

#### a. Experimental

There exist many experimental investigations which support the need for the DSD having shape similar to that described in this work. Among these are the work of Joss and Gori (1978) and Gori and Geotis (1981) which have already been referred to in the previous section. Others which are especially important in the present context are the experimental measurements of Mueller (1965), Caton (1966), Blanchard (1953) and Dingle and Hardy (1962).

The drop size spectra of Mueller were collected with a drop camera having a sampling volume of about  $1 \text{ m}^3$  and correspond to a variety of rainfall types including continuous rain, showers and thunderstorm rain, but not including orographic rain. Every one of Mueller's drop size spectra is concave downward indicating that they are described by a DSD having  $\mu > 0$ .

The experimental results of Caton were obtained from Doppler radar measurements at different heights above the surface in continuous rainfall. Although the sampling volume is much larger in this case (which would imply a greater degree of spatial averaging than is the case in Mueller's measurements), almost all of Caton's drop spectra are similar to Mueller's in that they can also be described by a gamma distribution with positive  $\mu$ . In addition, there is clear evidence in Caton's work that the shape of the DSD is dependent on altitude.

Some of the measurements of Blanchard (1953) were obtained in orographically induced rainfall and involve sampling volumes somewhat smaller than those of Ca-

ton and Mueller. Nevertheless, Blanchard's results indicate a clear tendency for increased numbers of small diameter drops in this type of rainfall which suggests that they could be described by a gamma DSD with  $\mu < 0$ .

Finally, the results of Dingle and Hardy (1962) are important since they offer insight into some of the physical mechanisms giving rise to changes in DSD shape. Their measurements, taken with a photoelectric spectrometer, show concave upward behavior for the DSD at the leading edges of showers and concave downward behavior in the rainshafts of thunderstorms. (They refer to the former behavior as a deficit and the latter as an excess of medium size drops relative to an exponential distribution). Dingle and Hardy attribute the concave upward behavior to the size sorting effects of gravity and wind shear at the leading edge of a shower. The concave downward behavior they conclude is due to the combined effects of breakup of large drops induced by gustiness and turbulence and the evaporation of small drops in the dry air beneath a thunderstorm.

*b. Theoretical*

In addition to the experimental evidence given here there exists an abundance of theoretical work which displays DSD shape effects similar to that considered in this work. Some of these investigations include those due to Hardy (1963), Mason and Ramanadham (1954), Ogura and Takahashi (1973), Brazier-Smith *et al.* (1972), List and Gillespie (1976) and Borchers *et al.* (1981). The first three of these assume that the modification of the DSD occurs at least partly within a supersaturated environment of cloud droplets, whereas the last three consider the drops to fall in unsaturated air. Each of these investigations involves some or all of the following physical mechanisms: condensation, accretion of cloud droplets, evaporation, collision-coalescence, collisional breakup and aerodynamic breakup. Obviously the first two mechanisms would not be included in a theoretical model of an unsaturated environment. In such a case the third and fourth mechanisms would tend to reduce the numbers of small drops, whereas the last two would affect a reduction in the numbers of large drops relative to some initial DSD.

As a specific example of the latter process, if it is assumed that the DSD is exponential initially at cloud base and that the drops fall in unsaturated air, then the effect of the above mechanisms would be to transform the DSD to concave downward shape as shown in Fig. 5a. Such a modification of the DSD is very clear in the work of Borchers *et al.* (1981) which is the most realistic of these theoretical calculations for unsaturated air since it includes all of the above physical mechanisms. In addition, the changes in DSD shape

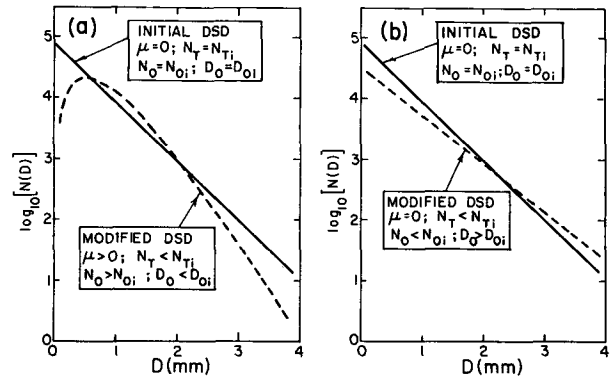


FIG. 5. (a) Changes in the raindrop size distribution as a result of evaporation, collision-coalescence, collisional breakup and aerodynamic breakup of raindrops in an unsaturated environment. The modification is assumed to occur at constant liquid water content so that the DSD parameters  $\mu$ ,  $N_0$  and  $D_0$  and the total number concentration of drops  $N_T$  change as shown on the figure. (b) Changes in the drop size distribution which would occur at constant liquid water content if it is assumed that the distribution must remain exponential. Other details as in (a).

with distance below cloud base as predicted by this theoretical work are in accord with the observations of Caton (1966) described earlier.

These changes in the DSD can be expressed quantitatively if it is assumed that they take place at constant liquid water content  $W$ . Since Borchers *et al.* (1981) show that the loss of liquid water due to evaporation is important only for the smallest drops which contribute only a small amount to the total liquid water content, then this assumption will be sufficiently accurate for the present purposes. The total number concentration of drops  $N_T$  ( $m^{-3}$ ) given by

$$N_T = \int_0^\infty N(D)dD = \frac{\Gamma(1 + \mu)}{(3.67 + \mu)^{1+\mu}} N_0 D_0^{1+\mu} \quad (15)$$

when Eqs. (3) and (5) are used, can then be expressed in terms of  $W$  by eliminating  $D_0$  between Eq. (15) and Eq. (6) with  $P = W$ ,  $p = 3$  and  $a_p = 0.524 \text{ g cm}^{-3}$ , as indicated in Table 1. The result so obtained is

$$N_T = N_0^{3/(4+\mu)} \frac{\Gamma(1 + \mu)}{[\Gamma(4 + \mu)]^{(1+\mu)/(4+\mu)}} \left(\frac{6W}{\pi}\right)^{(1+\mu)/(4+\mu)} \quad (16)$$

In similar fashion the median volume diameter can be written in terms of  $W$  as

$$D_0 = (3.67 + \mu) \left[ \frac{6W}{\pi N_0 \Gamma(4 + \mu)} \right]^{1/(4+\mu)} \quad (17)$$

These equations imply that a reduction in  $N_T$  at constant  $W$  due to breakup and coalescence requires an increase in  $N_0$  and a decrease in  $D_0$  concomitant with an increase in  $\mu$ . That is, if the DSD were initially exponential with DSD parameters having initial values  $\mu = 0$ ,  $N_0 = N_{0i}$ ,  $D_0 = D_{0i}$ , and  $N_T = N_{Ti}$  then, after

modification such that  $N_T < N_{Ti}$  and  $\mu > 0$ , one finds  $N_0 > N_{0i}$  and  $D_0 < D_{0i}$  as shown in Fig. 5a. The changes in DSD shape predicted by theory could not have been made consistent with these results if it were assumed that the DSD must remain exponential ( $\mu = 0$ ). In such a case a decrease in  $N_T$  at constant  $W$  would require that  $N_0 < N_{0i}$  and  $D_0 > D_{0i}$  so that the DSD is transformed to a form like that shown in Fig. 5b. Finally, it should be noted that the same kinds of changes in DSD shape are predicted in the first three of the above cited theoretical investigations where at least part of the trajectory of the drops is through a supersaturated cloud environment. Consequently, the description of the DSD in terms of a gamma distribution would be useful even in those cases where the total rainfall water content is not conserved.

In summary then, there is a wealth of experimental and theoretical work which demonstrates that the use of a two parameter exponential distribution is inadequate for describing the changes in the DSD which occur naturally in the atmosphere. The experimental evidence is based on thousands of observations from a wide variety of geographical locations and involves many different types of measuring instruments acquiring data aloft as well as at the surface of the earth. This large volume of data ensures that the DSD shape effects observed experimentally cannot be the result of measurement errors. Furthermore, the raindrop interactions inherent in theoretical models of DSD evolution provide additional support for concluding that DSD shape effects are real and are the result of the action of known physical mechanisms.

#### 4. Applications of the gamma drop size distribution

##### a. Empirical relations between integral parameters

The theoretical expressions developed earlier in this work are used in this section to deduce relationships between pairs of integral parameters. The results so obtained are compared with empirically derived relations of other investigations to determine the variations in  $N_0$  and  $\mu$  which occur in nature.

Let the first of the pair of integral parameters be designated  $P$  and written as in Eq. (7). If the second is designated  $Q$  then

$$Q = a_Q \frac{\Gamma(q + \mu + 1)}{(3.67 + \mu)^{q + \mu + 1}} N_0 D_0^{q + \mu + 1}. \quad (18)$$

Elimination of  $D_0$  between Eqs. (7) and (18) results in the form

$$P = \alpha Q^\beta \quad (19)$$

with

$$\beta = \frac{p + \mu + 1}{q + \mu + 1}, \quad (20)$$

$$\alpha = \frac{a_P \Gamma(p + \mu + 1) N_0^{1 - \beta}}{[a_Q \Gamma(q + \mu + 1)]^\beta}. \quad (21)$$

These equations can be inverted easily to find  $\mu$  and  $N_0$  in terms of  $\alpha$  and  $\beta$ . The results obtained are

$$\mu = \frac{p - \beta q}{\beta - 1} - 1, \quad (22)$$

$$N_0 = \left[ \frac{\alpha \left[ a_Q \Gamma\left(\frac{p - q}{\beta - 1}\right) \right]^\beta}{a_P \Gamma\left(\frac{\beta(p - q)}{\beta - 1}\right)} \right]^{1/(1 - \beta)} \quad (23)$$

For an empirical relation involving a given pair of integral parameters  $P$  and  $Q$ , substitution into Eq. (22)–(23) of the values of  $\alpha$  and  $\beta$  found from empirical analyses of rainfall data for  $P$  and  $Q$  yields the values of  $\mu$  and  $N_0$  which this empirical relation implies. Using the five integral parameters  $Z$ ,  $R$ ,  $W$ ,  $A$ , and  $\Sigma$  in Eq. (19) it is possible to derive a total of 10 independent relations between pairs of them. There exist in the meteorological literature empirical relations corresponding to virtually all of these possibilities. The most popular is of course the relation between  $Z$  and  $R$  for which  $P = Z$ ,  $p = 6$ ,  $a_P = 10^6 \text{ mm}^6 \text{ cm}^{-6}$  and  $Q = R$ ,  $q = 3.67$ ,  $a_Q = 33.31 \text{ mm h}^{-1} \text{ m}^3 \text{ cm}^{-3.67}$  as shown in Table 1. For this case Eq. (19) gives  $\beta = (7 + \mu)/(4.67 + \mu)$  which implies that the use of an exponential distribution ( $\mu = 0$ ) in this procedure predicts  $\beta = 7/4.67 = 1.5$ , the *only* value of  $\beta$  permitted by the assumption of such a form for the DSD. However, it is clear from the 69  $Z$ – $R$  relations listed by Battan (1973) that there is considerable variation in  $\beta$  and in  $\alpha$  between empirical  $Z$ – $R$  relations for different rainfall types as well as between different  $Z$ – $R$  relations for the same rainfall type. Ulbrich and Atlas (1978) show that these variations in  $\alpha$  and  $\beta$  are due to real physical differences between the types of rainfall to which the  $Z$ – $R$  relations apply. In addition, Wilson and Brandes (1979) consider each of the physical mechanisms enumerated in the previous section and show how the values of  $\alpha$  and  $\beta$  would be affected by the action of each mechanism. The observed variations in  $\alpha$  and  $\beta$  are therefore not due to measurement errors nor are they induced by correlations between the errors involved in measuring  $Z$  and  $R$ . The latter conclusion is especially appropriate in those cases where the measurements of the two integral parameters are made by different instruments. Such is the case in the work of Seliga *et al.* (1981), Desautels and Gunn (1970) and Wilson and Brandes (1979) which demonstrate considerable variation in  $\alpha$  and  $\beta$  in both space and time.

The values of  $\mu$  and  $N_0$  implied by the 69  $Z$ – $R$  relations of Battan (1973) have been calculated following the procedure given above and the numerical results for some of them are shown in Table 2. Only those  $Z$ – $R$  relations for which it is possible to clearly identify the type of rainfall to which the empirical

TABLE 2. Values of  $\alpha$  and  $\beta$  in  $Z = \alpha R^\beta$  found from empirical analyses of rainfall data, the values of the parameters  $\mu$  and  $N_0$  in the gamma DSD implied by these relations, and the corresponding values of  $\epsilon$  and  $\delta$  in the implied  $D_0$ - $R$  relation of the form  $D_0 = \epsilon R^\delta$ .

Source	$\alpha$	$\beta$	$\mu$	$N_0$	$\epsilon$	$\delta$
Orographic rain:						
Wexler (1948)	208	1.53	-0.27	$4.27 \times 10^4$	0.080	0.23
Ramana Murty and Gupta (1959)	109	1.64	-1.03	$9.82 \times 10^3$	0.055	0.28
Blanchard (1953)	31	1.71	-1.39	$1.59 \times 10^4$	0.031	0.31
Thunderstorm rain:						
Jones (1956)	486	1.37	1.63	$2.05 \times 10^6$	0.130	0.16
Blanchard (1953)	290	1.41	1.01	$1.24 \times 10^6$	0.101	0.18
Savaramakrishnan (1961)	219	1.41	1.01	$2.46 \times 10^6$	0.090	0.18
Fujiwara (1965)	450	1.46	0.40	$7.05 \times 10^4$	0.118	0.20
Widespread or stratiform rain:						
Jones (1956)	313	1.25	4.65	$6.40 \times 10^{10}$	0.114	0.11
Atlas and Chmela (1957)	255	1.41	1.01	$7.53 \times 10^5$	0.110	0.18
Fujiwara (1965)	205	1.48	0.18	$1.96 \times 10^5$	0.082	0.21
Marshall and Palmer (1948)	220	1.60	-0.79	$7.24 \times 10^3$	0.077	0.26
Showers:						
Jones (1956)	380	1.24	5.04	$9.20 \times 10^{10}$	0.129	0.10
Fujiwara (1965)	300	1.37	1.63	$7.54 \times 10^6$	0.106	0.16
Imai (1960)	200	1.50	-0.01	$1.09 \times 10^5$	0.081	0.22
Muchnik (1961)	204	1.70	-1.34	$1.31 \times 10^3$	0.069	0.30
Foote (1966)	520	1.81	-1.79	$9.13 \times 10^1$	0.095	0.35
Higgs (1952)	127	2.87	-3.42	1.29	0.013	0.80

relation applies have been included in this table. In addition, none has been included for which the analysis leading to the equation implies either a specific form for the DSD or a relationship between any of the DSD parameters. By categorizing these relations in this way it is seen that  $\mu < 0$  for orographic rain (indicating a broad DSD with large numbers of small diameter drops) and  $0 < \mu < 2$  for thunderstorm rain (corresponding to a narrower DSD with reduced numbers of small drops). For widespread or stratiform rain the values of  $\mu$  are more variable but tend to be positive. For showers  $\mu$  is even more variable and no general statement may be made about its range of values.

### b. Empirical $D_0$ - $R$ relations

Further evidence of DSD shape effects can be found in the many extant empirically determined relations between  $D_0$  and  $R$ . A theoretical relation of this type can be deduced from Eq. (7) for the rainfall rate  $R$  (i.e.,  $P = R$  with  $a_p$  and  $p$  as given in Table 1). Solving the resultant equation for  $D_0$ , we get an expression of the form

$$D_0 = \epsilon R^\delta, \quad (24)$$

where

$$\epsilon = (3.67 + \mu)[33.31N_0\Gamma(4.67 + \mu)]^{-1/(4.67+\mu)}, \quad (25)$$

$$\delta = \frac{1}{4.67 + \mu}. \quad (26)$$

The results for  $\epsilon$  and  $\delta$  found by substituting into Eqs. (25) and (26) the values of  $N_0$  and  $\mu$  implied by the above  $Z$ - $R$  relations are listed in Table 2. The variation

of  $\epsilon$  and  $\delta$  with rainfall type is exactly as described by Atlas (1964) and by Atlas and Chmela (1957). That is, low  $\epsilon$  and high  $\delta$  are characteristic of orographic rain with small  $D_0$  and fallspeed, whereas higher  $\epsilon$  and lower  $\delta$  are representative of thunderstorm rain with larger drops. In addition, the ranges of values of  $\epsilon$  and  $\delta$  shown in Table 2 are similar to those given by Battan (1973). If the unusual results of Higgs (1952) are not included, these ranges are more than a factor of 4 for  $\epsilon$  and a factor of 3 for  $\delta$ . If the results of Higgs (1952) are included, these ranges are even larger. These variations could not be accounted for through the use of an exponential distribution.

### c. $N_0$ - $\mu$ relations

One of the interesting aspects of the results listed in Table 2 is the systematic tendency for  $N_0$  to increase as  $\mu$  increases. This is more clearly illustrated in Fig. 6 where  $\log_{10}N_0$  is plotted versus  $\mu$  as open circles for all of the 69  $Z$ - $R$  relations listed by Battan (1973). Although more than 85% of the data points lie in the range  $-2 \leq \mu \leq 3$ , these results show that there is a well-defined exponential dependence of  $N_0$  on  $\mu$  implied by these empirical relations over a wide range of values of  $\mu$ . The same analysis as that described above for the  $Z$ - $R$  relations has been performed for several other empirical relations between the pairs of integral parameters  $Z$ - $W$ ,  $W$ - $A$ ,  $\Sigma$ - $R$ ,  $W$ - $R$ ,  $\Sigma$ - $W$  and  $A$ - $R$ . The results found from these relations are plotted in Fig. 6. They suggest the same relationship between  $N_0$  and  $\mu$  as implied by the  $Z$ - $R$  relations. Since some of these empirical relations are not dependent on the



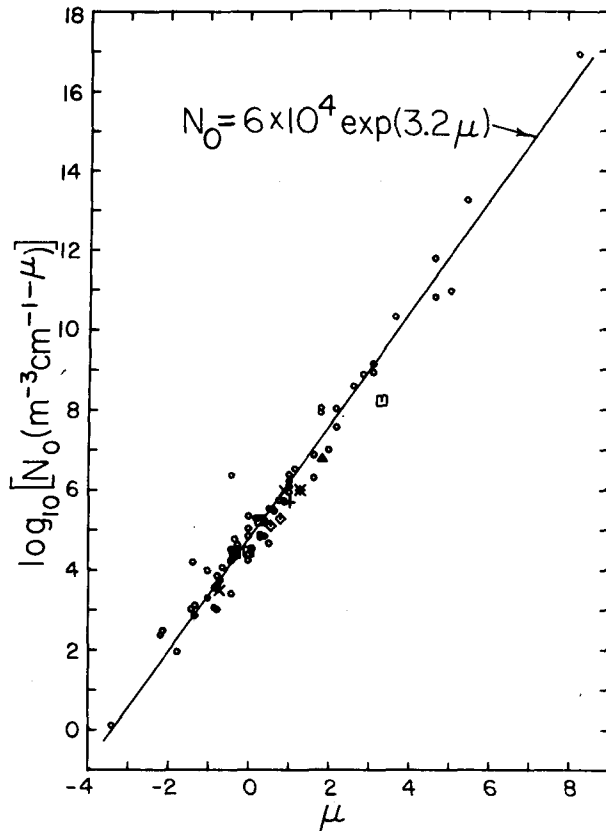


FIG. 6. Plot of the gamma DSD parameters  $N_0$  versus  $\mu$  as deduced from empirical relations between rainfall integral parameters. The solid circles are points found from the 69 empirical  $Z$ - $R$  relations listed by Battan (1973). The squares correspond to the empirical  $Z$ - $W$  relations of Marshall and Palmer (1948), Sivaramakrishnan (1961), Jones (1956) and Ulbrich and Atlas (1977). The triangle is a point found from the empirical  $W$ - $A$  relation of Eccles and Mueller (1971). The diamonds are found from the empirical  $\Sigma$ - $R$  relations of Atlas (1953) and Ulbrich and Atlas (1977). The crosses represent the empirical  $W$ - $R$  relations of Marshall and Palmer (1948) and Sivaramakrishnan (1961). The asterisk corresponds to the  $\Sigma$ - $W$  relation of Ulbrich and Atlas (1977) and the plus sign is found from the  $A$ - $R$  relation of Atlas and Ulbrich (1977) for a radar wavelength of 3.22 cm and a temperature of 10°C. The straight line and the equation shown on the figure represent a least squares fit to all the data.

drop fallspeed, the similarity of the results suggests that the  $N_0$ - $\mu$  relationship is not strongly dependent on the assumption of a power law dependence of drop fallspeed on diameter.

A linear least squares fit of  $\ln N_0$  versus  $\mu$  to all the data in Fig. 6 produces an expression of the form

$$N_0 = 6 \times 10^4 \exp(3.2\mu) \text{ [m}^{-3} \text{ cm}^{-1} \text{ } \mu\text{]} \quad (27)$$

with the linear correlation coefficient between  $\ln N_0$  and  $\mu$  for these data greater than 0.98. This very high correlation is not surprising in view of the dependence of  $N_0$  on  $\mu$  implied theoretically by Eq. (23). This dependence is shown in Fig. 7 where  $\log_{10} N_0$  is plotted versus  $\mu$  with  $P = Z$ ,  $Q = R$ , and with  $\alpha$  as a parameter having values equal to 100, 200, 300 and 500  $\text{mm}^{6-\beta}$

$\text{m}^{-3} h^\beta$ . Since more than 75% of the empirical values of  $\alpha$  fall in the range  $150 \leq \alpha \leq 450 \text{ mm}^{6-\beta} \text{ m}^{-3} h^\beta$  and the  $N_0$ - $\mu$  relation is fairly insensitive to changes in  $\alpha$  within this range, then it is seen that Eq. (27) has both theoretical and empirical justification.

A similar relationship between  $N_0$  and  $\mu$  is also inherent in raindrop spectra from moment to moment within a given rainfall type. To demonstrate that such is the case, a set of experimental drop size spectra has been used to find the DSD parameters. A description of these data and the details concerning the method of collection can be found in Ulbrich and Atlas (1977). The method by which  $N_0$ ,  $\mu$  and  $\Lambda$  were found for these data first uses Eq. (9) to find  $\mu$  from the value of  $G'$  calculated from the size spectrum. This result is then used with the experimental value of  $D_m$  to find  $\Lambda$  (or  $D_0$ ) from Eq. (10). These two parameters are then used to find  $N_0$  from Eq. (6) with  $P = W$ , the liquid water content. The fits of Eq. (2) to the experimental data which result from this procedure are better than those found by least squares in the sense that they predict values of integral parameters such as  $R$  and  $Z$  which are closer to those found directly from the experimental data. This finding is probably due to the fact that this procedure uses three integral parameters ( $G'$ ,  $D_m$ ,  $W$ ) to find the DSD parameters ( $N_0$ ,  $D_0$ ,  $\mu$ ) which are in turn used to find the values of

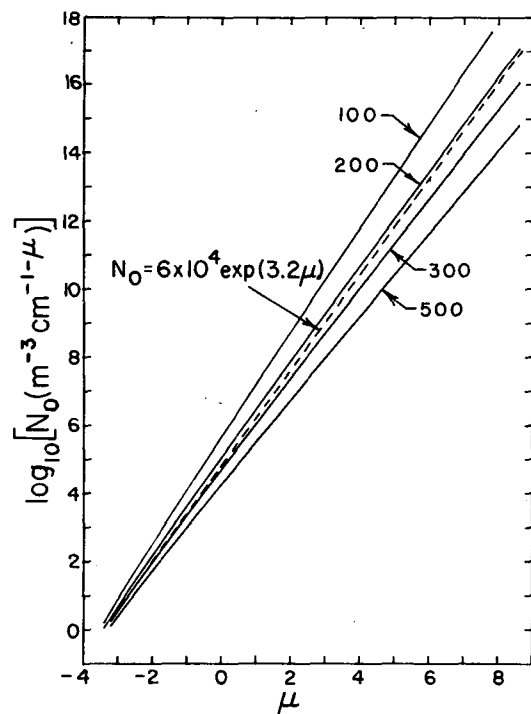


FIG. 7. Theoretical dependence of  $N_0$  on  $\mu$  with  $\alpha$  as a parameter as implied by empirical  $Z$ - $R$  relations of the form  $Z = \alpha R^\beta$ . Each curve is labeled with the value of  $\alpha$  to which it corresponds. Also shown as a dashed line is the least squares  $N_0$ - $\mu$  line from Fig. 6.

other integral parameters. This result was also noted in the work of Ulbrich and Atlas (1977) and is a consequence of the greater weighting of the large drops implicit in the calculation of the DSD parameters from  $G'$ ,  $D_m$ , and  $W$  as compared with that embodied in a traditional semilogarithmic least squares fit.

The values of  $N_0$  and  $\mu$  found by the above procedure for the set of experimental drop size spectra used in this work are plotted in Fig. 8 where it is seen that the experimental spectra display the same kind of tendency for  $N_0$  to increase as  $\mu$  increases as found from the empirical  $Z$ - $R$  relations. However, the  $N_0$ - $\mu$  relation shown in Fig. 6 and plotted in Fig. 8 as the solid straight line is seen to generally overestimate the results found from the experimental spectra. To display this in quantitative terms a linear least squares fit of  $\ln N_0$  versus  $\mu$  to the data in Fig. 8 yields the result

$$N_0 = 1.52 \times 10^4 \exp(3.14\mu) \text{ [m}^{-3} \text{ cm}^{-1-\mu}] \quad (28)$$

with a correlation coefficient greater than 0.95. This relation is plotted in Fig. 8 as the dashed straight line. Although the constant in the argument of the exponential function in Eq. (28) is close to that in Eq. (27), the coefficient is about a factor of four less. The origin of this difference is not known at present, but will be investigated in future work. The difference may be due

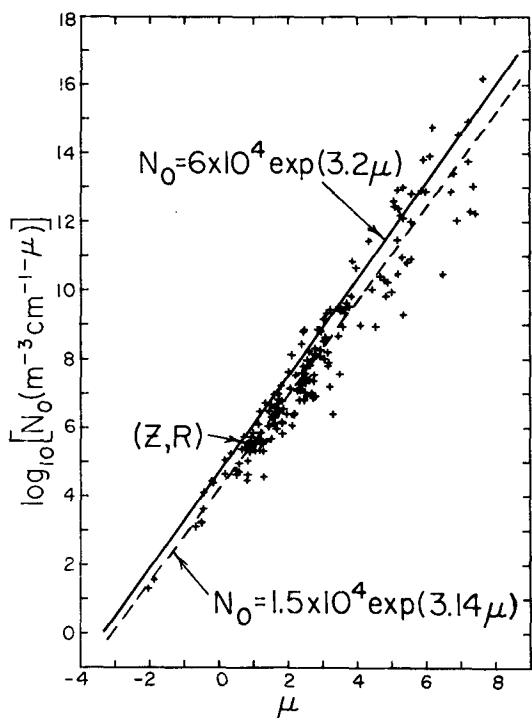


FIG. 8. Plot of the gamma DSD parameters  $N_0$  versus  $\mu$  as deduced from experimental drop size spectra from moment to moment within a given rainfall type. The dashed line is a least squares fit to all the data. The solid straight line is the least squares  $N_0$ - $\mu$  line from Fig. 6. Also shown as  $(Z, R)$  and as a solid circle is the point deduced from the empirical  $Z$ - $R$  relation which applies to these data.

to the fact that most of the empirical relations used to find the data plotted in Fig. 6 involve a greater degree of spatial averaging than the data shown in Fig. 8. An alternative explanation might be that the results in Fig. 6 were derived from a pair of integral parameters ( $Z$  and  $R$ ) which weight the large diameter end of the size spectrum more heavily than the parameters ( $G'$ ,  $D_m$ ,  $W$ ) used to find the results in Fig. 8. To substantiate this latter point use can be made of the empirical  $Z$ - $R$  analysis performed by Ulbrich and Atlas (1977) who find  $Z = 366R^{1.42}$  for the size spectrum data used in this work. When this relation is used to find  $N_0$  and  $\mu$  by the procedure described earlier, it produces a datum on the  $N_0$ - $\mu$  diagram in Fig. 8 shown by the solid circle. Although this point lies in the center of the range of  $\mu$ 's within which the overwhelming majority of the points in Fig. 6 lie, it does not fall in the center of the data in Fig. 8 (where 75% of the data fall in the range  $-1 \leq \mu \leq 4$ ). Further work will be required using spectral data for other rainfall events to determine whether the difference between these two  $N_0$ - $\mu$  relations is generally valid.

The equations deduced in the previous section relating the total number concentration  $N_T$  and the median volume diameter  $D_0$  to the liquid water content  $W$  may be used to illustrate the physical significance of the  $N_0$ - $\mu$  relation found in this work. If Eq. (27) is used in place of  $N_0$  in Eqs. (16) and (17) for  $N_T$  and  $D_0$ , respectively, and it is assumed that  $W$  is constant, then it is easy to show from these equations that the effect of increasing  $\mu$  is to decrease  $N_T$  and  $D_0$  when  $W > 0.1 \text{ g m}^{-3}$  whereas when  $W < 0.1 \text{ g m}^{-3}$ ,  $N_T$  decreases and  $D_0$  increases when  $\mu$  increases. If this relationship between  $N_0$  and  $\mu$  did not exist and these two parameters were allowed to vary independently, then it would be possible to find combinations of them for which an increase in  $\mu$  would produce an increase in  $N_T$ . Consequently, use of the  $N_0$ - $\mu$  relation found in this work ensures that the changes in DSD shape produced by increasing  $\mu$  will be accompanied by changes in  $N_T$  consistent with that which has been shown in the previous section to be predicted theoretically.

### 5. Implications in terms of dual-measurement techniques

The existence of an  $N_0$ - $\mu$  relation of the form found in the previous section has implications in rainfall measurement techniques which use pairs of remote measurables. One of the first of these proposed techniques involves the pair of remote measurables reflectivity factor  $Z$  and microwave attenuation  $A$ . This specific example can be used as an illustration of how the results found in this work are incorporated into dual-measurement methods. The expressions for  $Z$  and  $A$  found from Eq. (6) and Table 1 can be combined by elimination of  $D_0$  to yield

$$Z = \xi(\mu) + \frac{10(7 + \mu)}{(5 + \mu)} \log_{10} A, \quad (29)$$

where

$$\xi(\mu) = 10 \log_{10} \left[ \frac{10^6 \Gamma(7 + \mu) N_0^{-2/(5+\mu)}}{[0.434 C_A \Gamma(5 + \mu)]^{(7+\mu)/(5+\mu)}} \right] \quad (30)$$

and the units of  $Z$  and  $A$  are dBZ and dB km<sup>-1</sup>, respectively. For simplicity it has been assumed in this example that  $n = 4$  in Table 1 for the definition of  $A$ , which according to Atlas and Ulbrich (1974) would apply to an attenuating radar wavelength  $\lambda \approx 1.5$  cm and requires  $C_A = 14.6$  cm<sup>-2</sup>.

If Eq. (27) is used in Eq. (30) for  $N_0$ , then  $\xi(\mu)$  is a function of  $\mu$  only. Consequently, for given  $Z$  and  $A$  Eq. (27) represents an equation for  $\mu$ , the solution to which is then combined with the theoretical expression for either  $Z$  or  $A$  found from Eq. (6) to find  $D_0$ . All other integral rainfall parameters follow directly from this solution.

Ulbrich (1981) has shown that rainfall parameters deduced from this particular combination of remote measurables are not very sensitive to DSD effects. A dual-measurement method in which DSD variations are more important is that illustrated by Ulbrich and Atlas (1977) and which involves  $Z$  and  $\Sigma$ . The incorporation of the  $N_0$ - $\mu$  relation found in this work into this method follows exactly the same approach as that given above for  $Z$  and  $A$ . However, for various reasons both of these methods have profound problems associated with actual implementation in the field. The most successful field-implemented technique to date is the differential reflectivity ( $Z_{DR}$ ) technique of Seliga and Bringi (1976) which Ulbrich and Atlas (1983) have shown is also the most sensitive of the methods to the DSD effects considered in this work. In fact they find that the use of an exponential distribution in the  $Z_{DR}$  method systematically overestimates the rainfall rate by almost 30%. However, Ulbrich (1983) has shown that the theoretical calculation of  $Z_{DR}$  which uses the gamma DSD can be combined with the corresponding calculation of the reflectivity factor at horizontal polarization ( $Z_H$ ) with the factor  $N_0$  which occurs in  $Z_H$  expressed in terms of  $\mu$ . The resultant  $Z_{DR}$ - $Z_H$  relationship is a function of  $\mu$  only (when the variation of raindrop axial ratio with equivalent spherical diameter is assumed to be known). This relationship can therefore be used with experimentally determined values of  $Z_{DR}$  and  $Z_H$  to find  $\mu$  for that observation. The result so obtained can be combined with  $Z_{DR}$  and  $Z_H$  to find  $N_0$  and  $D_0$  from which all integral parameters of interest follow directly provided that the limits of integration can be specified [as in Eq. (6)]. Ulbrich (1983) finds that the difference between the rainfall rate deduced by this method and the actual rainfall rate is small and only a few percent larger than that which results when three remote measurables are employed.

## 6. Conclusions

It may be concluded from this work that the use of a three-parameter raindrop size distribution in analyses of rainfall data describes very well the variation in distribution shape which is observed experimentally in nature and predicted theoretically. Although a gamma distribution of the form  $N(D) = N_0 D^\mu \exp(-\Lambda D)$  has been used here for the raindrop size distribution in arriving at the conclusion, it is not the only possible form and others may serve equally well in describing these natural variations in DSD shape. However, the gamma distribution is especially convenient since it produces simple expressions for integral rainfall parameters and is a generalization of the exponential form found by Marshall and Palmer (1948).

It has also been found theoretically and empirically that the parameters  $N_0$  and  $\mu$  are related through an approximate expression of the form  $N_0 = C_N \exp(3.2\mu)$  where  $C_N$  is a constant. This form appears to apply to the systematic variations of  $N_0$  with  $\mu$  within a given rainfall type as well as between different rainfall types. However, the coefficient  $C_N$  has the value  $6 \times 10^4$  m<sup>-3</sup> cm<sup>-1- $\mu$</sup>  when found from empirical relations between integral parameters for different rainfall types and the value  $1.5 \times 10^4$  m<sup>-3</sup> cm<sup>-1- $\mu$</sup>  when found from analysis of raindrop size spectra from moment to moment within a given rainfall. More work needs to be done with additional rainfall data to establish whether the difference between these two results for  $C_N$  is generally valid.

The existence of a relationship between  $N_0$  and  $\mu$  implies that not all three of the gamma size distribution parameters are independent. Equivalently, it may be stated that the size distribution can be reduced to and is adequately represented by a two parameter form but that this form is not exponential. This means that dual-measurement techniques can employ the gamma distribution to obtain estimates of rainfall parameters if the parameter  $N_0$  is expressed in terms of  $\mu$  or vice versa. Examples of this approach to dual-measurement techniques have been considered in this work.

Finally, the results of this work have implications concerning the structure of the rain parameter diagram of Ulbrich and Atlas (1978). In future work an investigation will be made of the effects of assuming a gamma drop size distribution on this graphical depiction of the relationships between rainfall parameters.

*Acknowledgment.* This work was supported by Grant ATM-8120686 from the National Science Foundation.

## REFERENCES

- Atlas, D., 1953: Optical extinction by rainfall. *J. Appl. Meteor.*, **10**, 486.
- , 1964: Advances in radar meteorology. *Advances in Geophysics*, Vol. 10, 317-478.
- , and A. C. Chmela, 1957: Physical-synoptic variations of drop-size parameters. *Proc. Sixth Weather Radar Conf.*, Boston, Amer. Meteor. Soc., pp. 21-30.

- , and C. W. Ulbrich, 1974: The physical basis for attenuation-rainfall relationships and the measurement of rainfall parameters by combined attenuation and radar methods. *J. Rech. Atmos.*, **8**, 275–298.
- , and —, 1977: Path- and area-integrated rainfall measurement by microwave attenuation in the 1–3 cm band. *J. Appl. Meteor.*, **16**, 1322–1331.
- , —, and R. Meneghini, 1984: The multi-parameter remote measurement of rainfall. *Radio Sci.* (in press).
- Bartnoff, S., and D. Atlas, 1951: Microwave determination of particle size distribution. *J. Meteor.*, **8**, 130–131.
- Battani, L. J., 1973: *Radar Observation of the Atmosphere*. University of Chicago Press, 324 pp.
- Blanchard, D. C., 1953: Raindrop size distribution in Hawaiian rains. *J. Meteor.*, **10**, 457–473.
- Borchers, R. G., P. Das and J. R. Scoggins, 1981: Modification of raindrop size distributions in subcloud downdrafts. *Preprints 20th Conf. on Radar Meteorology*, Boston, Amer. Meteor. Soc., pp. 392–396.
- Brazier-Smith, P. R., S. G. Jennings and J. Latham, 1972: The interaction of falling drops: Coalescence. *Proc. Roy. Soc. London*, **A326**, 393–408.
- Bringi, V. N., T. A. Seliga and W. A. Cooper, 1984: Analysis of aircraft hydrometeor spectra and differential reflectivity ( $Z_{DR}$ ) radar measurements during the Cooperative Convective Precipitation Experiment. *Radio Sci.* (in press).
- Caton, P. G. F., 1966: A study of raindrop size distributions in the free atmosphere. *Quart. J. Roy. Meteor. Soc.*, **92**, 15–30.
- Desautels, G., and K. L. S. Gunn, 1970: Comparison of radar with network rain gauges. *Preprints 14th Conf. on Radar Meteorology*, Tucson, Amer. Meteor. Soc., 239–240.
- Dingle, A. N., and K. R. Hardy, 1962: The description of rain by means of sequential rain-drop size distributions. *Quart. J. Roy. Meteor. Soc.*, **88**, 301–314.
- Donnadieu, G., 1982: Observation de deux changements des spectres des gouttes de pluie dans une averse de nuages stratiformes. *J. Rech. Atmos.*, **14**, 439–455.
- Eccles, P. J., and E. A. Mueller, 1971: X-band attenuation and water content estimation by a dual-wavelength radar. *J. Appl. Meteor.*, **10**, 1252–1259.
- Foote, G. B., 1966: A Z-R relation for mountain thunderstorms. *J. Appl. Meteor.*, **2**, 229–231.
- Fujiwara, M., 1965: Raindrop-size distribution from individual storms. *J. Atmos. Sci.*, **22**, 585–591.
- Goddard, J. W. F., and S. M. Cherry, 1984: The ability of dual-polarization radar (co-polar linear) to predict rainfall rate and microwave attenuation. *Radio Sci.* (in press).
- Goldhirsh, J., and I. Katz, 1974: Estimation of raindrop size distribution using multiple wavelength radar systems. *Radio Sci.*, **9**, 439–446.
- Gori, E. G., and S. G. Geotis, 1981: Comparison of raindrop size distributions observed in New England and Switzerland. *Preprints 20th Conf. on Radar Meteorology*, Boston, Amer. Meteor. Soc., 282–286.
- Hardy, K. R., 1963: The development of raindrop size distributions and implications related to the physics of precipitation. *J. Atmos. Sci.*, **20**, 299–312.
- Hauser, D., and P. Amayenc, 1983: Exponential size distributions of raindrops and vertical air motions deduced from vertically pointing-Doppler radar data using a new method. *J. Climate Appl. Meteor.*, **22**, 407–418.
- Higgs, A. J., 1952: The measurement of precipitation by radar. *Proc. Third Weather Radar Conf.*, McGill University, Montreal, Amer. Meteor. Soc., D49–D50.
- Imai, I., 1960: Raindrop size-distributions and Z-R relationships. *Proc. Eighth Weather Radar Conf.*, Boston, Amer. Meteor. Soc., 211–218.
- Jones, D. M. A., 1956: Raindrop-size distribution and radar reflectivity. Res. Rep. No. 6, Illinois State Water Survey, Meteor. Lab., Urbana, 20 pp.
- Joss, J., and E. G. Gori, 1978: Shapes of raindrop size distributions. *J. Appl. Meteor.*, **17**, 1054–1061.
- Khrigian, A. K. A., I. P. Mazin and V. Cao, 1952: Distribution of drops according to size in cloud. *Tr. Tsent. Aerol. Observ.*, **7** (in Russian), 56.
- Laws, J. O., and D. A. Parsons, 1943: The relation of raindrop-size to intensity. *Trans. Amer. Geophys. Union*, **24**, 452–460.
- Levin, L. M., 1961: Studies in the physics of coarsely dispersed aerosols. *Izv. Akad. Nauk. SSSR*, (in Russian).
- List, R., and J. R. Gillespie, 1976: Evolution of raindrop spectra with collision-induced breakup. *J. Atmos. Sci.*, **33**, 2007–2013.
- Marshall, J. S., and W. M. K. Palmer, 1948: The distribution of raindrops with size. *J. Meteor.*, **5**, 165–166.
- Mason, B. J., and R. Ramanadham, 1954: Modification of the size distribution of falling raindrops by coalescence. *Quart. J. Roy. Meteor. Soc.*, **80**, 388–394.
- Muchnik, V. M., 1961: The accuracy of radar rain intensity measurements. *Meteor. Gidrol.*, **2** (in Russian), 44–47.
- Mueller, E., 1965: Radar rainfall studies. Ph.D. dissertation, University of Illinois.
- Ogura, Y., and T. Takahashi, 1973: The development of warm rain in a cumulus model. *J. Atmos. Sci.*, **30**, 262–277.
- Ramana Murty, B. V., and S. C. Gupta, 1959: Precipitation characteristics based on raindrop size measurements at Delhi and Khandala during southwest monsoon. *J. Sci. Ind. Res.*, **18A**, 352–371.
- Sekhon, R. S., and R. C. Srivastava, 1970: Snow size spectra and radar reflectivity. *J. Atmos. Sci.*, **27**, 299–307.
- Seliga, T. A., and V. N. Bringi, 1976: Potential use of differential reflectivity measurements at orthogonal polarizations for measuring precipitation. *J. Appl. Meteor.*, **15**, 69–76.
- , —, and H. H. Al-Khatib, 1981: A preliminary study of comparative measurements of rainfall rate using the differential reflectivity radar technique and a raingauge network. *J. Appl. Meteor.*, **20**, 1362–1368.
- Sivaramakrishnan, M. V., 1961: Studies of raindrop size characteristics in different types of tropical rain using a simple raindrop recorder. *Indian J. Meteor. Hydrol. Geophys.*, **12**, 189–217.
- Sulakvelidze, G. K., 1969: *Rainstorms and Hail* (translated from Russian), Israel Program for Scientific Translations, Jerusalem [available from NTIS].
- , and Yu. A. Dadali, 1971: Multiwavelength radar measurements of precipitation intensity. *Radar Meteorology, Proc. Third All-Union Conf.* (translated from Russian), Israel Program for Scientific Translations, 32–45. [available from NTIS].
- Ulbrich, C. W., 1981: Effect of size distribution variations on precipitation parameters determined by dual-measurement techniques. *Proc. 20th Conf. on Radar Meteorology*, Boston, Amer. Meteor. Soc., 276–281.
- , 1983: An empirical method of accounting for variations in the form of the raindrop size distribution in dual-measurement techniques. *Preprints 21st Conf. Radar Meteorology*, Edmonton, Amer. Meteor. Soc., 317–322.
- , and D. Atlas, 1975: The use of radar reflectivity and microwave attenuation to obtain improved measurements of precipitation. *Proc. 16th Conf. on Radar Meteorology*, Houston, Amer. Meteor. Soc., 496–503.
- , and —, 1977: A method for measuring precipitation parameters using radar reflectivity and optical extinction. *Ann. Telecommun.*, **32**, 415–421.
- , and —, 1978: The rain parameter diagram: methods and applications. *J. Geophys. Res.*, **83**, 1319–1325.
- , and —, 1984: Assessment of the contribution of differential polarization to improved rainfall measurements. *Radio Sci.* (in press).
- Waldvogel, A., 1974: The  $N_0$  jump of raindrop spectra. *J. Atmos. Sci.*, **31**, 1067–1078.
- Weber, S. F., 1976: The  $D_m$  analysis of precipitation particle size distributions. Rep. 76-14, Inst. Atmos. Sci., S. Dakota School of Mines and Technology, 78 pp.
- Wexler, R., 1948: Rain intensities by radar. *J. Meteor.*, **5**, 171–173.
- Wilson, J. W., and E. A. Brandes, 1979: Radar measurement of rainfall—A summary. *Bull. Amer. Meteor. Soc.*, **60**, 1048–1058.