# NOTES AND CORRESPONDENCE

## The Equations of Motion for a Shallow Rotating Atmosphere and the "Traditional Approximation"

NORMAN A. PHILLIPS[1]

*National Center for Atmospheric Research, Boulder, Colo.*

20 April 1966

A long-standing point of confusion in atmospheric dynamics has been the role of the coriolis terms proportional to the cosine of the latitude. In hydrostatic problems they are ignored for very logical reasons, but, as pointed out by Eckart (1960), their significance is not so clear in non-hydrostatic problems. This note provides a rationale for discarding them in quite general motions of a *shallow* atmosphere.

Let $r$ (radius), $\lambda$ (longitude) and $\varphi$ (latitude) represent spherical coordinates. Because of the small ellipticity of the earth, these coordinates, with "sea-level" located at $r=a=$ constant, and "gravity" having no $\varphi$-component, are reasonably exact for analyzing atmospheric motions. The relative velocity components are

$$\left.\begin{aligned} u&=h_\lambda\frac{d\lambda}{dt}=r\cos\varphi\frac{d\lambda}{dt}\\ v&=h_\varphi\frac{d\varphi}{dt}=r\frac{d\varphi}{dt}\\ w&=h_r\frac{dr}{dt}=1\frac{dr}{dt}\end{aligned}\right\}, \tag{1}$$

while the equations of motion are usually written as

$$\left.\begin{aligned} \frac{du}{dt}&=F_\lambda+\left(2\Omega+\frac{u}{r\cos\varphi}\right)(v\sin\varphi-w\cos\varphi)\\ \frac{dv}{dt}&=F_\varphi-\left(2\Omega+\frac{u}{r\cos\varphi}\right)u\sin\varphi-\frac{wv}{r}\\ \frac{dw}{dt}&=F_r-g+\left(2\Omega+\frac{u}{r\cos\varphi}\right)u\cos\varphi+\frac{v^2}{r}\end{aligned}\right\}. \tag{2}$$

$F_\lambda$, $F_\varphi$ and $F_r$ are the three components of the pressure and frictional forces per unit mass, while $g$ represents both the Newtonian gravitational attraction of the earth and the centrifugal acceleration due to $\Omega$. These equations are not exact because they include the assumption that the ellipticity of the earth is small, so that near the surface of the earth geographic latitude differs only slightly from geocentric latitude. This note is not concerned with this approximation, however, but with a further simplification usually introduced into (2) for mathematical convenience. Nonetheless, it is important to note here that the approximations leading to (2) have not corrupted the angular momentum principle; (2) and (1) together imply that

$$\frac{d}{dt}[r\cos\varphi(u+\Omega r\cos\varphi)]=r\cos\varphi F_\lambda. \tag{3}$$

From this point on, (1) and (2) are considered as completely accurate.

The *shallowness* of the atmosphere (as far as meteorologists are concerned) certainly seems to justify the practice of replacing $r$ as a coefficient by $a$ (and $\partial/\partial r$ by $\partial/\partial z$) in the hydrodynamic equations (2). It appears that this has usually been done, either explicitly or implicitly, by simply replacing (1) and (2), respectively, by

$$\left.\begin{aligned} u'&=a\cos\varphi d\lambda/dt\\ v'&=a\,d\varphi/dt\\ w'&=dz/dt\end{aligned}\right\}, \tag{1'}$$

and

$$\left.\begin{aligned} \frac{du'}{dt}&=F_\lambda+\left(2\Omega+\frac{u'}{a\cos\varphi}\right)(v'\sin\varphi-w'\cos\varphi)\\ \frac{dv'}{dt}&=F_\varphi-\left(2\Omega+\frac{u'}{a\cos\varphi}\right)u'\sin\varphi-\frac{w'v'}{a}\\ \frac{dw'}{dt}&=F_z-g+\left(2\Omega+\frac{u'}{a\cos\varphi}\right)u'\cos\varphi+\frac{v'^2}{a}\end{aligned}\right\}. \tag{2'}$$

[See, for example, Haltiner and Martin (1957, pp. 166–168) and Eckart (1960, p. 100).]

These equations have the serious deficiency that they do not possess an angular momentum principle, i.e., there appears to be no function $A(\varphi,z)$ (other than zero) such that $AF_\lambda=dM/dt$ for arbitrary $\Omega$, $u'$, $v'$, $w'$, $F_\varphi$ and $F_z$. The difficulty is caused by the $w'$ terms on

[1] Permanent address: Department of Meteorology, Massachusetts Institute of Technology, Cambridge, Mass.

the right hand side of the first equation in (2′). *If these terms were absent*, it is easy to show that (2′) implies

$$\frac{d}{dt}[a\cos\varphi(u'+\Omega a\cos\varphi)]=a\cos\varphi F_\lambda, \qquad (3')$$

a logical approximation to (3).

This failure of (2′) arises because (2), into which the approximation $r=a$ has been introduced, is not in a vector invariant form. Such a form is

$$\frac{\partial \mathbf{v}}{\partial t}=\mathbf{F}+\mathbf{g}-\nabla(\tfrac{1}{2}\mathbf{v}^2)+\mathbf{v}\times\text{curl}(\mathbf{v}+\mathbf{v}_e). \qquad (4)$$

At the level of approximation contained in (2), $\mathbf{v}_e$ is a vector of magnitude $\Omega h_\lambda=\Omega r\cos\varphi$ directed eastward, and $\mathbf{g}$ has only a radial component.

We now introduce in *all* curvilinear operators the approximate scale factors

$$\left.\begin{aligned} h_\lambda'&=a\cos\varphi\\ h_\varphi'&=a\\ h_z'&=1\end{aligned}\right\}. \qquad (5)$$

The coordinates are not altered in their role as independent orthogonal Eulerian variables except by the purely notational change of replacing $r$ by $a+z$. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ represent the orthogonal unit vectors directed eastward, northward and upwards which are implied in both (2) and (2′). From (5) we obtain

$$\begin{aligned}\mathbf{v}&=u'\mathbf{i}+v'\mathbf{j}+w'\mathbf{k}\\ &=h_\lambda'\frac{d\lambda}{dt}\mathbf{i}+h_\varphi'\frac{d\varphi}{dt}\mathbf{j}+h_z'\frac{dz}{dt}\mathbf{k},\end{aligned} \qquad (6)$$

and

$$\nabla=\frac{\mathbf{i}}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{\mathbf{j}}{a}\frac{\partial}{\partial\varphi}+\mathbf{k}\frac{\partial}{\partial z}, \qquad (7)$$

$$\begin{aligned}\frac{d}{dt}&=\frac{\partial}{\partial t}+\frac{d\lambda}{dt}\frac{\partial}{\partial\lambda}+\frac{d\varphi}{dt}\frac{\partial}{\partial\varphi}+\frac{dz}{dt}\frac{\partial}{\partial z},\\ &=\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla,\end{aligned} \qquad (8)$$

as would, indeed, be understood in (2′). But we also have

$$\mathbf{v}_e=\Omega h_\lambda'\mathbf{i}=\Omega a\cos\varphi\,\mathbf{i}, \qquad (9)$$

$$\text{curl}\mathbf{v}=\left(\frac{1}{a}\frac{\partial w'}{\partial\varphi}-\frac{\partial v'}{\partial z}\right)\mathbf{i}+\left(\frac{\partial u'}{\partial z}-\frac{1}{a\cos\varphi}\frac{\partial w'}{\partial\lambda}\right)\mathbf{j}+\frac{1}{a\cos\varphi}\left(\frac{\partial v'}{\partial\lambda}-\frac{\partial u'\cos\varphi}{\partial\varphi}\right)\mathbf{k}, \qquad (10)$$

and, most significantly,

$$\text{curl}\mathbf{v}_e=2\Omega\sin\varphi\,\mathbf{k}. \qquad (11)$$

Eq. (4) can now be written out in its three components. For comparison with (2′), however, it is more convenient to then rewrite these in terms of $du'/dt$, etc., by making use of (8) and (10). The results are

$$\left.\begin{aligned}\frac{du'}{dt}&=F_\lambda+\left(2\Omega+\frac{u'}{a\cos\varphi}\right)v'\sin\varphi\\ \frac{dv'}{dt}&=F_\varphi-\left(2\Omega+\frac{u'}{a\cos\varphi}\right)u'\sin\varphi\\ \frac{dw'}{dt}&=F_z-g\end{aligned}\right\}. \qquad (2'')$$

These equations do satisfy (3′). The fact that they are simpler than (2′) is a pleasant result, but the more important fact from a theoretical standpoint is that their derivation is compatible with all curvilinear operators, and they therefore do not violate any fundamental physical principles.

Several final comments may be made.

1) The approximation (9) in principle should also be introduced into the definition of the geopotential and $g$. This modification, however, is no more significant than is the neglect of the earth's ellipticity already present in (2).

2) Approximation (11) implies that the full vorticity $2\Omega$ of the earth's rotation must be replaced by its vertical component if one wishes to approximate $r$ by $a$. Conversely, it would appear that if $2\Omega\mathbf{j}\cos\varphi$ is thought to be important in a particular hydrodynamic phenomenon in the atmosphere, the complete spherical equations (1) and (2) must be considered (at least as the original point of departure upon which further special approximations might be imposed).

3) The present argument evidently places in a clearer perspective the "traditional approximation" invoked by Eckart (1960, pp. 95-101) in beginning his analysis of atmospheric perturbations. It also explains to some extent why linearized versions of (2′) lead to unconventional mathematical problems.

4) Most treatments of atmospheric motions (such as large-scale numerical weather prediction and linearized analyses of the tidal problem) have also used the consistent equations (2″), but with the hydrostatic approximation in place of the last equation in (2″). The hydrostatic approximation is basically different than the geometric approximation (5); in the form $F_z=g$, it eliminates all inertial terms in the last equation of (2′). Conservation of energy is then enough to require elimination of the $w'$ terms in the first two equations of (2′). (The assumed smallness of $w'$ *vis-a-vis* $u'$ and $v'$ upon which the hydrostatic relation is based will also eliminate those terms in a scale analysis.) In the absence

of the hydrostatic approximation, conservation of energy alone is not enough to force (2′) into (2″).

5) It should perhaps be emphasized that the arguments leading to (2″) in place of (2′) do not necessarily provide any additional *justification* for ignoring the complete effects of sphericity. The validity of the approximations (5), in the final analysis, can be established only by how well deductions based on (2″) agree with observation.

REFERENCES

Eckart, C., 1960: *Hydrodynamics of Oceans and Atmospheres.* New York, Pergamon Press, 290 pp.

Haltiner, G., and F. Martin, 1957: *Dynamical and Physical Meteorology.* New York, McGraw-Hill, 470 pp.